

Derivatives of rotation number of one parameter families of circle diffeomorphisms

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ABSTRACT. We consider the rotation number $\rho(t)$ of a diffeomorphism $f_t = R_t \circ f$, where R_t is the rotation by t and f is an orientation preserving C^∞ diffeomorphism of the circle S^1 . We shall show that if $\rho(t)$ is irrational

$$\limsup_{t' \rightarrow t} (\rho(t') - \rho(t)) / (t' - t) \geq 1.$$

Dedicated to Professor Kazuo Masuda on his 65-th birthday

1. Introduction

Let f be an orientation preserving C^∞ diffeomorphism of the circle $S^1 = \mathbb{R}/\mathbb{Z}$, and consider a one parameter family f_t , $t \in J = [-1/2, 1/2]$, of diffeomorphisms defined by $f_t = R_t \circ f$, where R_t denotes the rotation by t . Fix once and for all a lift $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$ of f to the universal covering \mathbb{R} of S^1 . Then a lift \tilde{f}_t of f_t is chosen as $\tilde{f}_t = T_t \circ \tilde{f}$, where T_t is the translation by t . The rotation number $\rho(t) \in \mathbb{R}$ of \tilde{f}_t is a continuous and nondecreasing function of t . Define a closed set C by

$$C = J \setminus \text{Int}(\rho^{-1}(\mathbb{Q})),$$

and assume for simplicity that $\rho(-1/2) = 0$, $\rho(1/2) = 1$ and C is contained in the interior of J .

V. I. Arnold [A] showed that $m(C) > 0$, where m denotes the Lebesgue measure. Denote by \mathcal{N} the set of non Liouville numbers and define a Borel subset N of C by

$$N = \rho^{-1}(\mathcal{N}).$$

M. R. Herman [H1] showed that ρ is an absolutely continuous function and that $m(N) > 0$, (under a much less restrictive condition on the one parameter family). A famous theorem of J.-C. Yoccoz says that if $t \in N$, then f_t is C^∞ conjugate to a rotation. Thus the result of M. R. Herman says that the set of the value t such that f_t is C^∞ conjugate to a rotation has positive Lebesgue measure. On the other hand it is known ([H2] p.170 and [KH] p.412) that for a generic value t in C the conjugacy of f_t to a rotation is a non absolutely continuous homeomorphism, provided that f is a real analytic diffeomorphism and f' is not constantly equal to 1. Nevertheless

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it is shown that $m(C \setminus N) = 0$ ([**T**]) and furthermore that $\dim_H(C \setminus N) = 0$ ([**G**]), where \dim_H denotes the Hausdorff dimension. The purpose of this paper is to show a somewhat stronger result in this direction.

Theorem 1. *If $\rho(t)$ is irrational, then we have*

$$\limsup_{t' \rightarrow t} \frac{\rho(t') - \rho(t)}{t' - t} \geq 1.$$

Notice that the above theorem implies by the absolute continuity of ρ that $\rho^{-1}(B)$ is null if B is a null Borel set. As for the case $\rho(t)$ is rational, we have:

Theorem 2. *Assume that f is real analytic and f' is not constantly equal to 1. For $t \in C$ such that $\rho(t) \in \mathbb{Q}$, we have*

$$\limsup_{t' \rightarrow t} \frac{\rho(t') - \rho(t)}{t' - t} = \infty.$$

These phenomena can be found in the computer graphics of the derivative ρ' in [**LV**]. The plan of the paper is as follows. In Sect. 2, we prove a weaker version of Theorem 1 and apply it to a new proof of the result of [**G**]. In Sect. 3, we give an elaboration of the argument of Sect. 2, which yields a proof of Theorem 1 for $\rho(t)$ a Liouville number, while the non Liouville case is treated in Sect. 4. Finally Sect. 5 is devoted to the proof of Theorem 2. Also we shall remark that it is necessary to consider \limsup instead of \liminf in Theorem 1.

2. Weaker version of Theorem 1

The purpose of this section is to show the following proposition which is a weaker version of Theorem 1, and by applying it to prove that $\dim_H(C \setminus N) = 0$ ([**G**]).

A positive integer q is called a *closest return* of an irrational number α if for any $0 < j < q$, we have $|j\alpha|_{S^1} > |q\alpha|_{S^1}$, where $|x|_{S^1}$ is the distance of x and 0 in S^1 . If $|q\alpha|_{S^1} = |q\alpha - p|$ for some integer p , the rational number p/q is called a *convergent* of α .

Proposition 2.1. *Suppose $\rho(t)$ is irrational, p/q a convergent of $\rho(t)$, $t' \in \rho^{-1}(p/q)$ the point nearest to t . Then we have*

$$\frac{\rho(t') - \rho(t)}{t' - t} \geq e^{-M},$$

where $M = \|(\log f')'\|_{C^0}$.

To begin with let us prepare the following lemma.

Lemma 2.2. *If $\rho(0)$ is irrational, then for any nonnegative integers i, j , we have $\mu((f^i)' \circ f^j) \geq 1$, where μ is the unique f -invariant probability measure on S^1 .*

PROOF. By the downward concavity of \log , it suffices to show

$$\mu(\log(f^i)' \circ f^j) = 0.$$

Since μ is f -invariant, this is equivalent to

$$\mu(\log(f^i)') = 0.$$

Again since μ is f -invariant and

$$\log(f^i)' = \sum_{\nu=0}^{i-1} \log f' \circ f^\nu,$$

this follows from

$$\mu(\log f') = 0.$$

By the unique ergodicity of f , we have a uniform convergence

$$n^{-1} \log(f^n)' = n^{-1} \sum_{k=0}^{n-1} \log f' \circ f^k \rightarrow \mu(\log f') = a.$$

But if $a > 0$ and if n is sufficiently large we have

$$(f^n)' > \exp \frac{an}{2} > 1,$$

and if $a < 0$, then

$$(f^n)' < \exp \frac{an}{2} < 1.$$

In any case these contradict

$$\int_{S^1} (f^n)'(x) dx = 1.$$

□

PROOF OF PROPOSITION 2.1. Assume that t in Proposition 2.1 is 0. The rotation number of \tilde{f} , $\rho(0) = \alpha$, is irrational by the hypothesis.

Given $x, y \in S^1$, denote

$$[x, y] = \{z \in S^1 \mid x \preceq z \preceq y\},$$

where \preceq is the positive cyclic order of S^1 , and by $y - x$ the length of $[x, y]$.

Assume that p/q is a convergent of α , and, to fix the idea, that $q\alpha - p < 0$. Thus we have $\alpha < p/q$, and shall estimate the value of $t > 0$ such that $\rho(t) = p/q$.

Now since q is a closest return, the intervals $R_\alpha^j[0, -q\alpha + p]$, $0 \leq j \leq q-1$, are mutually disjoint, where R_α denotes the rotation by α . The diffeomorphism f is topologically conjugate to R_α by an orientation preserving homeomorphism which maps a given point x to 0. This implies that if we set $L(x) = [x, f^{-q}(x)]$, then

$$(2.1) \quad f^j L(x) \text{ are mutually disjoint for } 0 \leq j \leq q-1,$$

for any $x \in S^1$. Let t be the smallest positive value such that $\rho(t) = p/q$. Our aim is to estimate the value of $(p/q - \alpha)/t$ from below.

For $0 \leq s \leq t$ consider the point $f_s^q(x)$. For $s = 0$, this is just $f^q(x)$ and as $s \rightarrow t$ the point $f_s^q(x)$ increases from $f^q(x)$ towards $f_t^q(x)$ on the interval

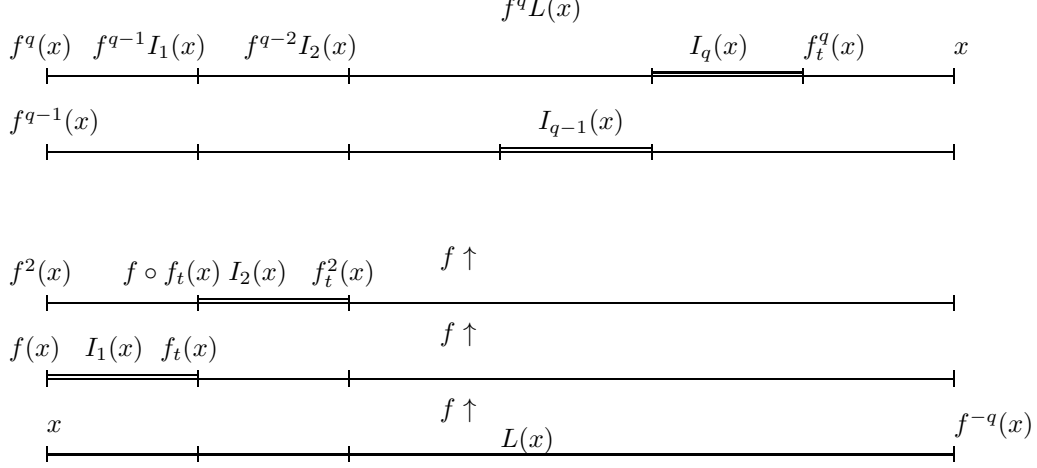
$$f^q L(x) = [f^q(x), x].$$

Thus for some x , $f_t^q(x) = x$, while for any x $f_t^q(x)$ lies on $f^q L(x)$, since t is the smallest value such that $\rho(t) = p/q$. This shows that the point $f_t^i(x)$ lies on the interval $f^i L(x)$ for $1 \leq i \leq q$. See the figure.

For each such i , consider the interval

$$I_i(x) = [f \circ f_t^{i-1}(x), f_t^i(x)] \subset f^i L(x).$$

Since $f_t^i(x) = f \circ f_t^{i-1}(x) + t$, these intervals have length t . Notice that the rightmost point of $I_i(x)$ is mapped by f to the leftmost point of $I_{i+1}(x)$.



The images

$$f^{q-1} I_1(x), f^{q-2} I_2(x) \cdots, f I_{q-1}(x), I_q(x)$$

form a sequence of consecutive intervals towards the right. Their union

$$\cup_{i=1}^q f^{q-i} I_i(x) = [f^q(x), f_t^q(x)]$$

is contained in $f^q L(x) = [f^q(x), x]$ for any x .

Let $\tau_i(x)$ be the length of $f^{q-i} I_i(x)$. By the Denjoy distortion lemma and (2.1), we get

$$\tau_i(x) \geq e^{-M} (f^{q-i})' \circ f^i(x) t,$$

where $M = \|(\log f')'\|_{C^0}$. Summing them up we get for any $x \in S^1$,

$$x - f^q(x) \geq e^{-M} t \sum_{i=1}^q (f^{q-i})' \circ f^i(x).$$

Now let us evaluate the both hand sides by the invariant measure μ of f . It is well known that the evaluation of the left term yields -1 times the rotation number of \tilde{f}^q , i. e.

$$\mu(\text{Id} - f^q) = p - q\alpha.$$

Therefore Lemma 2.2 implies that

$$(p/q - \alpha)/t \geq e^{-M},$$

as is desired. □

Now let us start the proof of Graczyk's Theorem ([G]). First we need the following easy lemma.

Lemma 2.3. *Assume α is irrational, $q > 1$, and for some $d > 3$*

$$|\alpha - p/q| < 1/q^d.$$

Then p/q is a convergent of α . □

Let

$$J_d(p/q) = \rho^{-1}((p/q - 1/q^d, p/q + 1/q^d)) - \rho^{-1}(p/q).$$

Corollary 2.4. *If $d > 3$, d is irrational and $q > 1$, then*

$$m(J_d(p/q)) \leq 2e^M q^{-d},$$

where m denotes the Lebesgue measure.

PROOF. Apply Proposition 2.1 to the irrational numbers $\alpha = p/q - 1/q^d$ and $\alpha = p/q + 1/q^d$. \square

Now the preimage L of the set of Liouville numbers by ρ can be described as

$$L = \bigcap_{d>3} \bigcap_{q_0} \bigcup_{q \geq q_0} J_d(p/q),$$

where in the union p runs over the integers $0 < p < q$, coprime to p . Given any $\alpha > 0$, if $d > 2/\alpha$, we have

$$\sum_{q \geq q_0} m(J_d(p/q))^\alpha \leq \sum_{q=q_0}^{\infty} 2^\alpha e^{\alpha M} q \cdot q^{-\alpha d} \rightarrow 0 \quad (q_0 \rightarrow \infty),$$

which concludes that $\dim_H(L) = 0$.

3. Proof of Theorem 1 for Liouville $\rho(t)$

Let q_n be the n -th closest return of the irrational number $\alpha = \rho(0)$. Then the sequence $\{q_n \alpha\}$ converges to 0 in S^1 , changing signs alternately. The closest returns satisfy

$$q_{n+1} = a_{n+1}q_n + q_{n-1},$$

where a_{n+1} is the $(n+1)$ -st denominator of the continued fraction of α .

Here we assume that α is a Liouville number. Thus the sequence $\{a_{n+1}\}$ is unbounded. It is no loss of generality to assume that there is a subsequence n_i such that

$$q_{n_i} \alpha \uparrow 0 \text{ in } S^1 \text{ and } a_{n_i+1} \rightarrow \infty.$$

For simplicity we shall write n_i as n in what follows, and have in mind that

$$q_n \alpha \prec 0 \prec -q_n \alpha \prec q_{n-1} \alpha$$

and that a_{n+1} is as large as desired. All the notations of the previous section are used by replacing p/q with p_n/q_n .

Consider the first return map S of the rotation R_α^{-1} on the interval $[q_n \alpha, q_{n-1} \alpha]$. We have

$$S = \begin{cases} R_\alpha^{-q_n} & \text{on } [q_n \alpha, (q_n + q_{n-1})\alpha] & \text{sending it to } [0, q_{n-1}\alpha] \\ R_\alpha^{-q_{n-1}} & \text{on } [(q_n + q_{n-1})\alpha, q_{n-1}\alpha] & \text{sending it to } [q_n \alpha, 0]. \end{cases}$$

Since there are ordering

$$0 \prec (1 - a_{n+1})q_n \alpha \prec (q_n + q_{n-1})\alpha \prec -a_{n+1}q_n \alpha \prec q_{n-1}\alpha,$$

the map $S = R_\alpha^{-q_n}$ sends the interval $[0, (1 - a_{n+1})q_n \alpha]$ onto $[-q_n \alpha, -a_{n+1}q_n \alpha]$. In particular $R_\alpha^{-\nu q_n}[0, -q_n \alpha]$, $0 \leq \nu < a_{n+1}$, form a consecutive sequence of intervals

contained in $[0, q_{n-1}\alpha]$. Translating into f via the topological conjugacy, we have for any $x \in S^1$

$$(3.1) \quad \bigcup_{\nu=0}^{a_{n+1}-1} f^{-\nu q_n} L(x) \subset K(x),$$

where $L(x) = [x, f^{-q_n}(x)]$ as before and $K(x) = [x, f^{q_{n-1}}(x)]$.

As is well known, easy to show, (2.1) can be extended to:

$$(3.2) \quad f^j K(x) \text{ are disjoint for } 0 \leq j \leq q_n - 1.$$

So it looks plausible that the total length $l(x)$ of $\bigcup_j f^j L(x)$ is very small, since a large number of its iterates by $f^{-\nu q_n}$ are mutually disjoint (except the end points), by virtue of (3.1) and (3.2). On the other hand the Denjoy distortion lemma actually guarantees that the coefficient e^{-M} in Proposition 2.1 can be replaced by e^{-Ml} where l is the maximum of $l(x)$, and hence if $l(x)$ were small enough, we should be able to prove Theorem 1. However we cannot do this for $L(x)$ itself and instead consider a subinterval $\hat{L}(x)$ defined by

$$\hat{L}(x) = [x, f^{-q_n} \circ f_t^{q_n}(x)],$$

where as before t is the smallest value such that $\rho(t) = p_n/q_n$. We are going to show that the total length of the union of intervals

$$\hat{l}(x) = m\left(\bigcup_{j=1}^{q_n} f^j \hat{L}(x)\right)$$

is small, where m denotes the Lebesgue measure as before. Notice that this is enough for our purpose of applying the Denjoy distortion lemma, that is, Proposition 2.1 can be improved as

$$(3.3) \quad \frac{\rho(t) - \rho(0)}{t} \geq e^{-M\hat{l}},$$

where $\hat{l} = \max\{\hat{l}(x) \mid x \in S^1\}$. Now we have

$$(3.4) \quad \hat{L}(x) = \bigcup_{i=1}^{q_n} f^{-i} I_i(x),$$

where as before

$$I_i(x) = [f \circ f_t^{i-1}(x), f_t^i(x)] \subset f^i L(x).$$

Put

$$\hat{l}_i(x) = m\left(\bigcup_{j=1}^{q_n} f^{j-i} I_i(x)\right).$$

Then we have by (3.4) and (2.1)

$$\hat{l}(x) = \sum_{i=1}^{q_n} \hat{l}_i(x).$$

Also (3.1) and (3.2) implies that

$$(3.5) \quad \sum_{\nu=0}^{a_{n+1}-1} \hat{l}(f^{-\nu q_n}(x)) \leq 1.$$

Let us compare $\hat{l}_i(x)$ with $\hat{l}_i(f^{-\nu q_n}(x))$ for $0 \leq \nu < a_{n+1}$. This is possible since the intervals $I_i(x)$ and $I_i(f^{-\nu q_n}(x))$ are contained in $f^i K(x)$ and of length t . In fact again by the Denjoy distortion lemma and (3.2), we get

$$(3.6) \quad m([f^{j-i} I_i(x)]) / t \leq e^N m([f^{j-i} I_i(f^{-\nu q_n}(x))]) / t$$

for any $0 < \nu < a_{n+1}$ and $1 \leq j \leq q_n$, where

$$N = \max\{\|(\log f')'\|_{C^0}, \|(\log(f^{-1})')'\|_{C^0}\}.$$

Summing up (3.6) by j , we get

$$\hat{l}_i(x) \leq e^N \hat{l}_i(f^{-\nu q_n} x).$$

Again summing up by i we obtain

$$\hat{l}(x) \leq e^N \hat{l}(f^{-\nu q_n}(x)).$$

Finally we get by (3.5)

$$a_{n+1} \hat{l}(x) \leq e^N \sum_{\nu=0}^{a_{n+1}-1} \hat{l}(f^{-\nu q_n}(x)) \leq e^N.$$

Now N is a constant depending only on f and a_{n+1} can be chosen arbitrarily large. By virtue of (3.3), this completes the proof of Theorem 1 for Liouville $\rho(t)$.

4. Proof of Theorem 1 for non-Liouville $\rho(t)$

Here we shall show that if $\rho(t)$ is non Liouville, then ρ is differentiable at t and $\rho'(t) \geq 1$.

In the first place we need the following theorem by P. Brunovský ([B]). For the proof see also [H1].

Theorem 4.1. *Let g_t be a C^1 -path of C^1 -diffeomorphisms such that g_t is an irrational rotation for some t . Then we have*

$$\frac{d}{dt} \text{rot}(g_t) = \int_{S^1} \frac{\partial g_t}{\partial t}(x) dx.$$

Since $\rho(t)$ is non Liouville, we have $f_t = h \circ R_{\rho(t)} \circ h^{-1}$ for some C^∞ diffeomorphism h ([Y]). Applying the Brunovský theorem to the family $h^{-1} \circ f_t \circ h$, we get

$$\begin{aligned} \rho'(t) &= \int_{S^1} \frac{\partial(h^{-1} \circ f_t \circ h)}{\partial t}(x) dx = \int_{S^1} (h^{-1})' \circ f_t \circ h(x) \frac{\partial f_t}{\partial t} \circ h(x) dx \\ &= \int_{S^1} (h^{-1})' \circ f_t \circ h(x) dx. \end{aligned}$$

Since $f_t \circ h = h \circ R_{\rho(t)}$ and the Lebesgue measure is invariant by the rotation we have

$$\rho'(t) = \int_{S^1} (h^{-1})' \circ h(x) dx = \int_{S^1} h'(x)^{-1} dx.$$

Now the Schwarz inequality concludes the proof of Theorem 1 for non-Liouville $\rho(t)$.

5. Proof of Theorem 2.

By the assumption of Theorem 2, for any $p/q \in \mathbb{Q}$, the set $\rho^{-1}(p/q)$ is a nondegenerate interval and C is a Cantor set. It is no loss of generality to assume that $0 \in J$ is the supremum of $\rho^{-1}(p/q)$ and to show

$$\lim_{t \downarrow 0} \frac{\rho(t) - (p/q)}{t} = \infty.$$

The real analyticity of f implies that the periodic points of $f_0 = f$ are finite in number, say x_ν , $1 \leq \nu \leq ql$. Now since 0 is the supremum of $\rho^{-1}(p/q)$, the graph of $\tilde{f}^q - p$ is above the diagonal and tangent to it just at the points $\pi^{-1}(x_\nu)$. For $\varepsilon > 0$, let $I_\nu = [a_\nu, b_\nu]$ be the ε -neighbourhood of x_ν . Choosing ε small enough, one can assume that the intervals I_ν are disjoint. Put $J_\nu = [b_\nu, a_{\nu+1}]$. Choose $N > 0$ big enough so that

$$f^{qj(\nu)}(b_\nu) \notin J_\nu, \quad 1 \leq j(\nu) \leq N+1.$$

This means that any orbit by f^q stays consecutively in J_ν for at most N times. Then since $\tilde{f}_t^q > \tilde{f}^q$ for $t > 0$ and the speed for f_t^q is bigger than that for f^q , any orbit by f_t^q stays consecutively in J_ν for at most N times.

On the other hand straightforward computation shows that $\partial f_t^q / \partial t \geq 1$, and therefore $\tilde{f}_t^q - p \geq t$. This shows that any orbit by f_t^q stays consecutively in I_ν for at most $2\varepsilon/t$ times, 2ε being the length of I_ν .

Let us estimate the times of iterations of f_t^q needed for some point to go around S^1 once. The above observation shows that the times needed for a round trip does not exceed $(N + 2\varepsilon/t)ql$. Translated into the language of rotation number we have

$$q\rho(t) - p \geq ((N + 2\varepsilon/t)ql)^{-1}.$$

Therefor if $t < \varepsilon/N$, we have

$$\frac{\rho(t) - (p/q)}{t} \geq (3\varepsilon q^2 l)^{-1},$$

completing the proof of Theorem 2.

Finally let us remark that taking \limsup instead of \liminf is necessary for Theorem 1.

Proposition 5.1. *Assume that f is real analytic and f' is not constantly equal to 1. There is a residual subset R in C such that for any $t \in R$*

$$\liminf_{t' \rightarrow t} \frac{\rho(t') - \rho(t)}{t' - t} = 0, \quad \text{and} \quad \limsup_{t' \rightarrow t} \frac{\rho(t') - \rho(t)}{t' - t} = \infty.$$

PROOF. Let $\rho(J) \cap \mathbb{Q} = \{\alpha_n \mid n \in \mathbb{N}\}$ and set $[a_n, b_n] = \rho^{-1}(\alpha_n)$. Then there is $c_n > b_n$ very near b_n such that if $t \in (b_n, c_n)$

$$\frac{\rho(t) - \alpha_n}{t - a_n} < \frac{1}{n} \quad \text{and} \quad \frac{\rho(t) - \alpha_n}{t - b_n} > n.$$

We used Theorem 2 for the second inequality. Now the set

$$R = \cap_p \cup_{n > p} (b_n, c_n) \cap C$$

is residual in C and satisfies the condition of the theorem. \square

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